MASTER

XXIX

A METHOD FOR CALCULATING THE ANGLE OF INCIDENCE OF THE DIRECT BEAM OF THE SUN ON A PLANE SURFACE OF ANY SLOPE AND ASPECT

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SUMMARY

By a suitable choice of co-ordinates, the cosine of the angle (θ) between the direct beam of the sun, and the normals to horizontal or sloping planes of any orientations may be expressed in the form

$$\cos\theta = A + B\cos t$$

where t is the hour angle from local apparent noon as apparent from the particular plane.

This permits the ready application of a simple graphical technique (Schutte (1931)).

Besides reducing very appreciably the extensive calculations necessitated by earlier methods, the more direct geometrical approach adopted renders the treatment of certain hitherto troublesome cases (e.g. the underside of sloping planes) relatively straightforward.

1. INTRODUCTION

An essential requirement for certain studies in "meso-" and "micro-" climate, is a method of estimating the radiation from sun and sky intercepted by plane surfaces of various slopes and aspects. This requires, inter alia, the calculation of the angle of incidence of the sun's beam on the surface. Various methods have been described in the literature e.g. by Brooks F.A. (1951), Carruthers N. (1948), Kaempfert W. (1942), Von Schedler A. (1950), Stagg J.M. (1950), Unna P.J. (1947), and an approximate graphical approach, in a publication of the Housing and Home Finance Agency, Washington (1954). It is now proposed to present a further method, which exhibits certain advantages of geometrical elegance and ease of calculation over the other techniques, and which permits the general application of a useful graphical device described by Schutte (1931).

After this work had been completed, the writer's attention was drawn to a paper published by Schutte (1943) in which the essentials of the method to be discussed had been set out. However, Schutte used a different coordinate system (viz. the celestial sphere) and did not explicitly show how the graphical device mentioned in his 1931 paper could be employed; in addition, it can be argued that the alternative approach now to be described reveals more clearly the geometrical significance of certain expressions.

The customary assumptions for this type of problem have been made, viz. the earth is spherical, the difference between solar and siderial time may be neglected, refraction can be ignored, and the sun's declination may be regarded as constant during any particular solar day. This last assumption is not absolutely essential, although without it Schutte's (1931) very useful technique cannot be applied.

2. METHOD

(a) The "corresponding" plane.

See Fig. 1.

If at any point S there is a direct south-facing slope of β to the horizontal then, following Unna (1947) and Carruthers (1948), a point T can be found of angular distance β southwards along the meridian where a horizontal plane will be parallel to the sloping plane at S. More generally if at S there is (as in Fig.1) a plane of slope β and "aspect" angle α (α is the angle between the vertical plane

through the line of greatest slope and the plane of the meridian through S) then T will lie arcal distance β along the great circle in the plane of the line of greatest slope at S.

Points such as T, T'. and tangent planes at T and T'.. will, following Schütte (1943), be termed "corresponding" points and "corresponding" planes respectively. The normals to the pairs of planes at S and T (or S' and T') will be parallel, making the angles θ (or θ') with the sun's beam. If I_0 be the intensity of the solar beam, the flux normal to the sloping surface will be $I_0 \cos\theta$; we therefore require $\cos\theta$ in terms of α, β, declination δ, and hour angle t.

(b) Co-ordinate System.

With O the centre of the earth as origin, choose a right-handed set of axes, Ox (towards the sun at the equinox), Oz (through pole P) and Oy. Suppose Ox remains fixed in space, then the line MO from the sun to O will lie in the xOz plane for all δ. α ($-180^\circ \leq \alpha \leq 180^\circ$) is measured positive from south through east and β ($0 \leq \beta \leq 90^\circ$) is the inclination of the line of greatest slope with the horizontal ($\beta = 0.$).

For the horizontal surface at S, the required angle $\theta(S)$ is $\angle$ MOS: and for the sloping surface it is $\theta(T)$ viz. $\angle$ MOT.

Local apparent time, defined by hour angle t, will be negative before local apparent noon (l.a.n.) and positive after l.a.n.

T is defined by its co-latitude (PT) and ε - the difference in longitude between S and T. Clearly for the sloping plane at S, l.a.n. as "apparent from the sloping plane" will occur at $\varepsilon/15$ hours before l.a.n. for the horizontal plane at S and $\varepsilon/15$ after in the case typified by S' and T'.

(c) Basic Formula

In the spherical triangle PQT, if ϕ be the latitude of S and ψ ($-90-\phi$) its colatitude then at hour angle t and declination δ

$$\cos\theta(T) = \sin\delta \cos(PT) + \cos\delta \sin(PT) \cos(\xi + \varepsilon) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where, from triangle PST,

$$\cos(PT) = \cos\psi \cos\beta + \sin\psi \sin\beta \cos(180-\alpha) \quad \ldots\ldots\ldots\ldots\ldots (2)$$

and

$$\cos\varepsilon = \frac{\cos\beta - \cos(PT)\cos\psi}{\sin(PT)\sin\psi} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The sign of ε is the same as for α (viz. positive for slopes with aspect from south through east to north).

When $\beta = 0$, Eq (1) reduces to the familiar form

$$\cos\theta(S) = \sin\delta \sin\phi + \cos\delta \cos\phi \cos t \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

In Eq (1) the quantities (PT) and ε are parameters defined by the geometry of the surface, whilst t and δ vary with time. The explicit occurrence of δ allows variations of $\theta(T)$ with time of year for a given surface to be readily studied: in contrast, the examination of variations of $\theta(T)$ for a given date but for different α and β may be more readily approached by von Schedler's formulation in which α and β occur explicitly.

For given ϕ and constant δ

Eq.(4) may be written $\cos\theta(S) = a + b \cos t$ $\quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$

and Eq. (1) $\cos\theta(T) = a' = b' \cos t'$ $\quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$

t' being the time from l.a.n. with respect to the sloping plane. Most advantages of this present formulation are direct consequences of the equivalence in form of Eqs. (5) and (6).

(d) Graphical technique of Schutte

Schutte (1931) noted that by plotting on cos/cos graph paper Eqs. 5 and 6 reduce to the form y = a + bx.

In practice these straight lines are most conveniently defined by cos t = 1; cos t' $\equiv$ cos (t + ε) = 1 at l.a.n. and cos θ(S) = cos θ(T) = 0 at apparent sunrise or sunset.

3. Illustrative Examples

(a) Let ϕ = 55°19'N. ie. ψ = 34 42'.

α = 45° β = 30° δ = 9°46'N.

i.e. a south-easterly slope of 30° at Eskdalemuir Observatory in Scotland in mid-April.

The under surface of this slope will be facing north-west i.e. will have a north-westerly aspect, and the appropriate values of cos θ for this case are given immediately by negative values from Eq.1.

Now α = + 45° thus ε is also positive.

From Eq.(2) cos (PT) = .51096; and from Eq. (3) cos ε = .91156

or ε = 24°17'.

Hence, cos θ (T) = .08668 + .84710 cos (t + ε) (from Eq. 1).

and cos θ(S) = .13950 + .56079 cos t. (from Eq. 4).

When t' = 0 t = - ε = -24°17' cos θ(T) = .93378 .....(7)

t = 0 cos θ(S) = .70235 .....(8)

Again when cos θ(T) = 0 t' $\equiv$ t + ε = $\pm$ 95°52' .....(9)

when cos θ(S) = 0 t = $\pm$ 104°24' .....(10)

The two lines are drawn in Figure 2.

See Fig. 2.

In Fig.3, cos θ(S) and cos θ(T) are related to local apparent time at S. Typical points on the "T" curve are:-

cos θ(T) = 0; t = - 24°17' - 95°52' = - 120°9'

or t = - 24°17' + 95°52' = + 71°35'

giving the points (a) and (c). If cos θ(T) = .500 then cos t' = cos (t + ε) = .490 (from Fig.2)

t' = $\pm$ 60°38', t = -84°55' or 36°21'

See Fig.3

In practice cos θ(T) and cos θ(S) would be read off at each half-hour l.a.t., as the data on radiation, sunshine and other elements likely to be studied in association with solar altitude are usually expressed in terms of quantities between exact hours.

(b) Let ϕ = 55°19'N, α = - 150°, β = 45°, δ = 23½°N.

i.e. a slope of 45° to the horizontal and aspect 30° west of north at the summer solstice.

$\alpha = -150^\circ$ therefore ε will be negative

$\cos(PT) = .92993$ and $\cos \varepsilon = -.27512$ or $\varepsilon = -105^\circ 58'$

Thus $\cos \theta(T) = .37081 + .33734 \cos t'$, and when $t' = 0$ we have $t = -\varepsilon = +105^\circ 58'$ and $\cos \theta(T) = .70815$. This, the maximum value of $\cos \theta(T)$, will occur at about 18.45 l.a.t. (i.e. $1200 + 105^\circ 58'/15$)

For $\cos \theta(T) = 0$ we have $\cos (t \pm \varepsilon) = -.37081/.33734$ which is inadmissible (see below), and for our second point we may choose $t = 90^\circ$,

then $t + \varepsilon = -15^\circ 58'$ and $\cos \theta(T) = .69514$

Some features require comment

(i) The "corresponding" point, of colatitude $21^\circ 35'$N, lies within the Arctic circle, thus when $\delta = 23\frac{1}{2}$N the sun will remain above the horizon. Hence a negative value of $\cos \theta(T)$ will not occur.

(ii) If $\cos \theta(T)$ is put equal to zero an inadmissible value of $\cos t'$ follows: another point must be chosen, this is conveniently that for $t = 90^\circ$.

(iii) When using Eq.(2) we need to write

$\cos (-180 - \alpha)$ giving $\cos (-180 + 150)$, since westerly aspects are measured by a negative rotation from south (see Fig.1).

(c) The final examples, illustrated in Fig. 4 are again for latitude $55^\circ 19'$N. and $\delta = 23\frac{1}{2}$N, when $\alpha = 0$ or 180 and $\beta = 0, 30^\circ, 90^\circ$.

The Schütte diagrams for all these cases may be superimposed, as the horizontal co-ordinates are all measured from the same zero.

In these cases there is, of course, no need to use Eq. (1) to obtain the maximum ordinate as this follows from simple geometrical considerations (see Fig.1). At the equinox (when $\delta = 0$) all lines will pass through the points on the abscissa corresponding to $t = 90$ or $\cos t = 0$. For other declinations we may either use Eq. (1) or, from tables, look up the local time of sunrise and sunset for places whose latitude is that of the "corresponding" point T.

Detailed comment is unnecessary.

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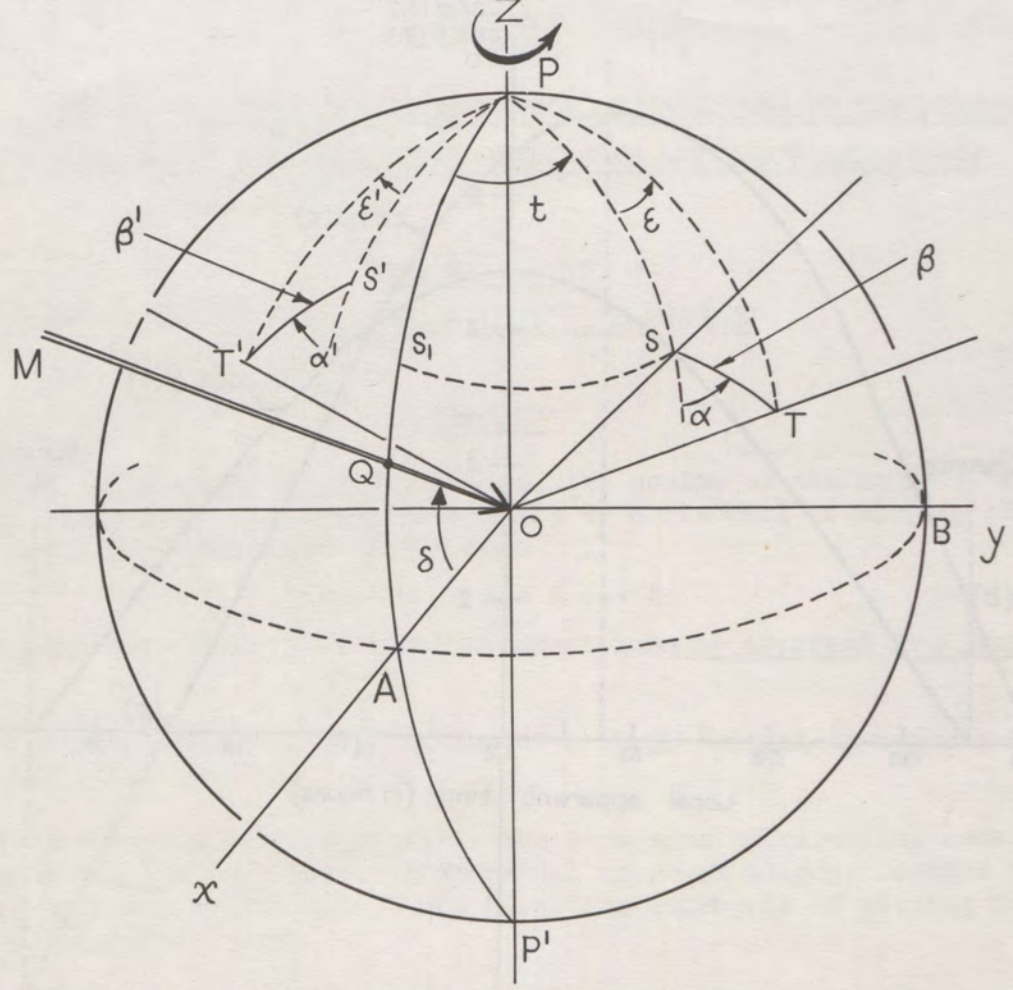

Figure 1. Co-ordinate system for calculation of angle between sun's ray (MQO) and OT, OT' parallel to the normals to surfaces at S, S' of slope β, β' and aspect angle α, α'.

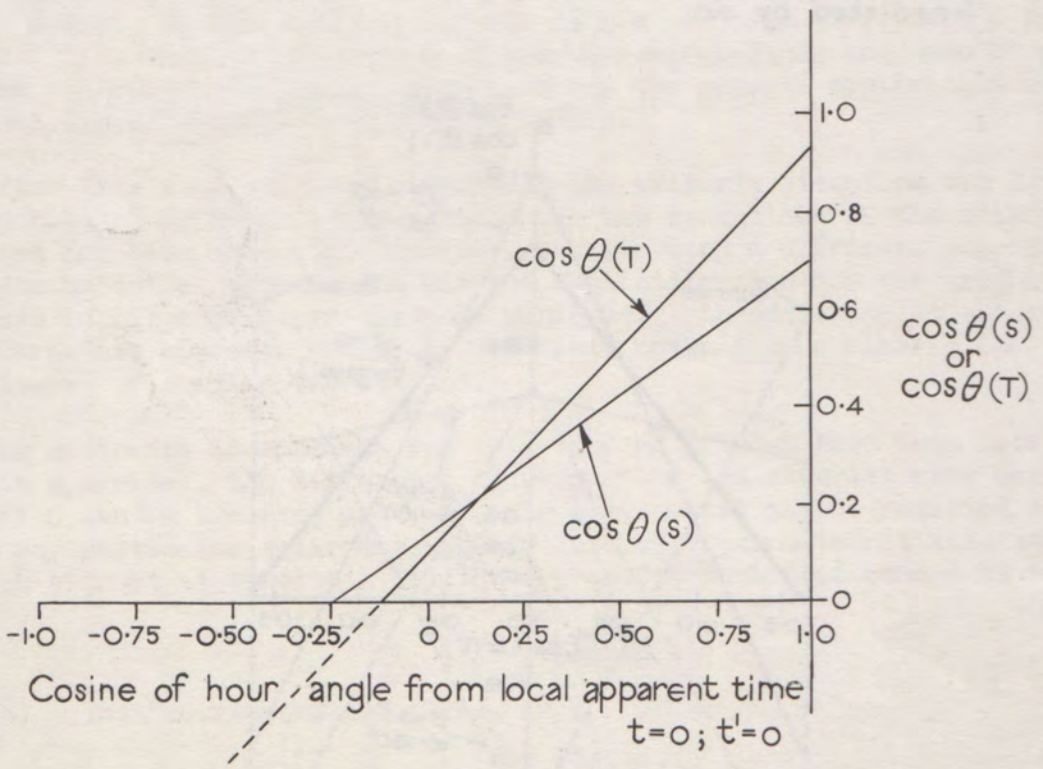

Figure 2. Cos θ(S) and cos θ(T) against cos t and cos t' respectively where θ(S), θ(T) are the angles between the sun's beam and normals to the horizontal and sloping surfaces; and t and t' (= t + ε) local apparent time with respect to the horizontal and the sloping surface.

Eskdalmuir Observatory (55° 19'N. 3° 12'W.) Solar declination 9° 46'N

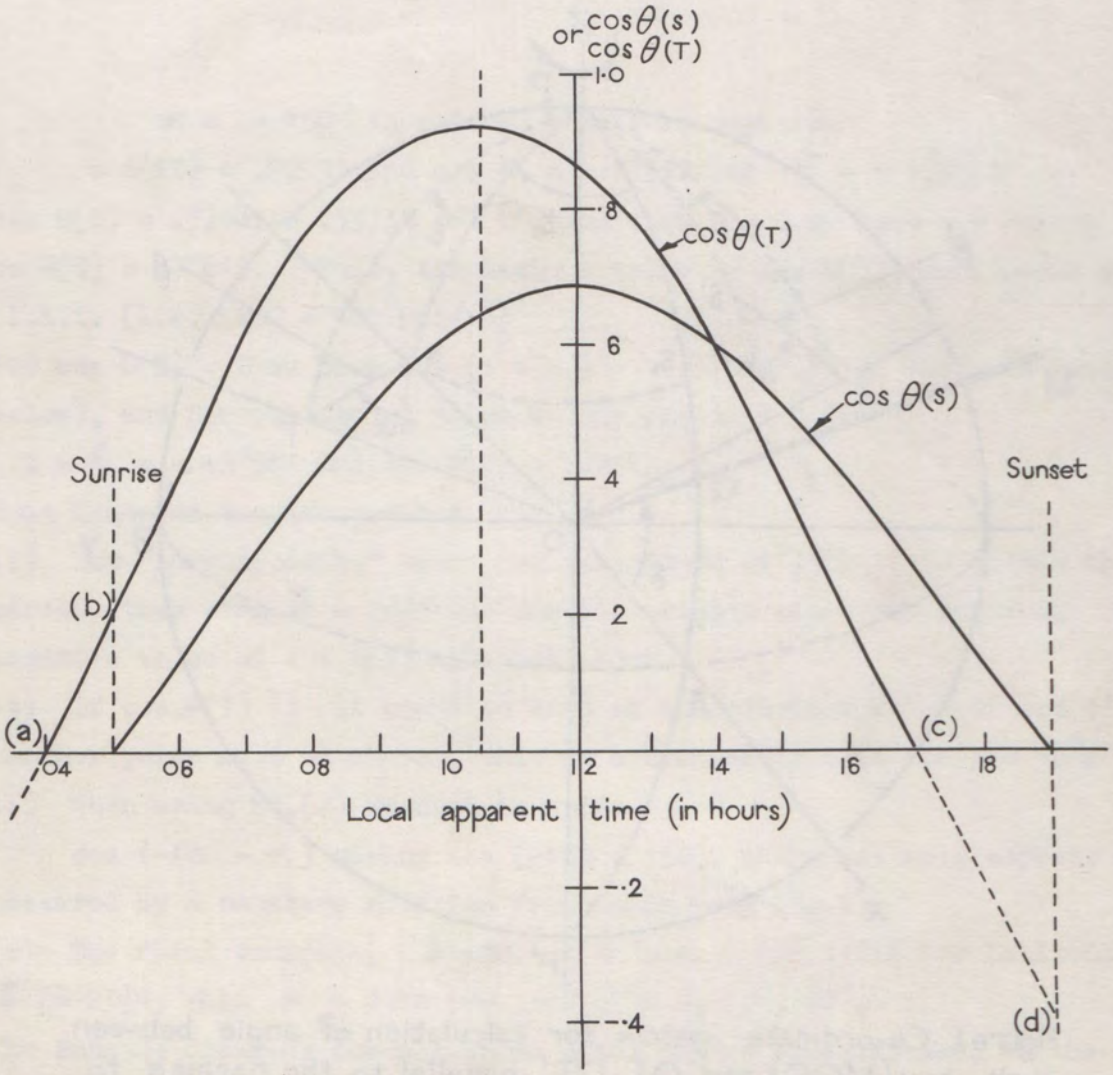

Figure 3 Cos θ(S) and cos θ(T) against local apparent time with respect to the horizontal surface, estimated graphically from Fig.2.

(a)(b) Sloping surface not irradiated by sun portion(a)(b) occurs before surrise

(c)(d) Negative value of cos θ(T) underside of surface irradiated by sun

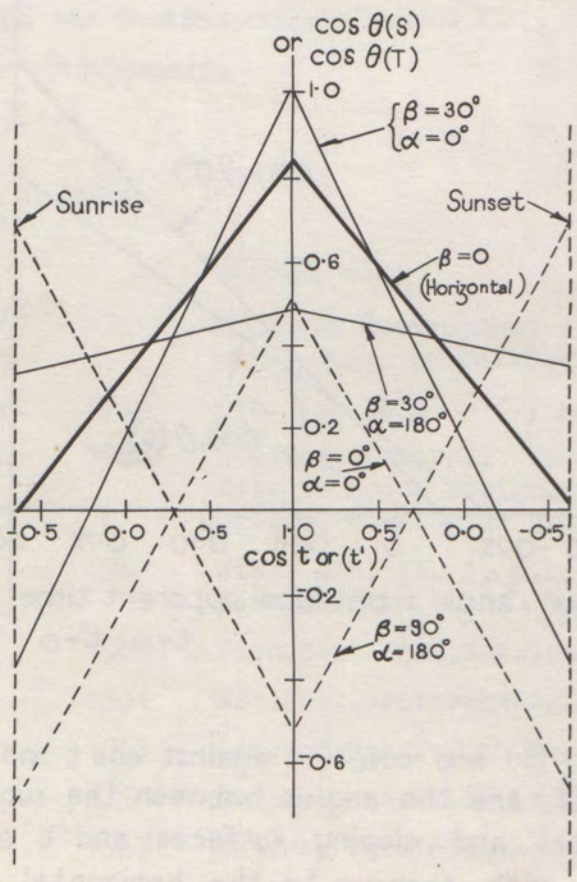

Figure 4. Valves of cos θ for horizontal (β=0) and for north and south facing slopes (β=30,90: α=0, 180) at 55° 19' N 3° 12' W. δ = 23½° N. (Surimer Solstice)